CALCULATING THE GREEKS BY CUBATURE FORMULAS

JOSEF TEICHMANN

ABSTRACT. We provide cubature formulas for the calculation of derivatives of expected values in the spririt of Terry Lyons and Nicolas Victoir. In financial mathematics derivatives of option prices with respect to initial values, so called Greeks, are of particular importance as hedging parameters. The proof of existence of Cubature formulas for Greeks is based on an argument, which leads to the calculation of Greeks in an asymptotic sense – even without Hörmander's condition. Cubature formulas then allow to calculate these quantities very quickly. Simple examples are added to the theoretical exposition.

1. Introduction

Cubature formulas provide approximative values for integrals with respect to a given measure. The well-developed theory of cubature formulas in finite dimensions was recently applied to provide Cubature formulas on Wiener space by Terry Lyons and Nicolas Victoir in the beautiful seminal article [8].

We try to extend the framework of Terry Lyons and Nicolas Victoir to the calculation of Greeks. We briefly outline in the introduction the main results and possible applications of it. In the sequel we shall always work with C^∞-bounded vector fields V_i and C^∞-bounded function f. Given a stochastic differential equation in $\mathbb{R}^N$ of the type

$$\begin{aligned} dY_t^y &= V_0(V_t^y)dt + \sum_{i=1}^d V_i(Y_t^y) \circ dB_t^i, \\ Y_0^y &= y, \end{aligned}$$

on a stochastic basis $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ with d-dimensional Brownian motion, then cubature formulas provide a method to approximate $E(f(Y_t^y))$, namely – choosing a degree m of accuracy – there is number $r \geq 1$ and there are H^1-trajectories $\omega_j : [0,T] \to \mathbb{R}^{d+1}$ and weights $\lambda_j > 0$, $j = 1,\dots,r$, such that

$$E(f(Y_t^y)) = \sum_{j=1}^r \lambda_j f(Y_t^y(\omega_j)) + \mathcal{O}(t^{\frac{m+1}{2}}),$$

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where $Y_t^y(\omega_j)$ denotes the point in $\mathbb{R}^N$, which is obtained by solving the ordinary, non-autonomous differential equation

$$\frac{dY_t^y(\omega_j)}{dt} = \sum_{i=0}^{d} V_i(Y_t^y(\omega_j))\frac{d\omega_i^i}{dt},$$
$$Y_0^y(\omega_j) = y.$$

The constant in the order estimate depends in general on derivatives up to order $m+1$, in the hypo-ellitptic case this dependence can be improved to the first derivative. Note also the important fact that the number r of trajectories can be bounded by a number depending on d and m, but not on N (see [8] for the precise estimate). Obviously this procedure can be iterated with small time steps by the Markov property, which yields then a high order numerical approximation scheme for the calculation of expected values $E(f(Y_t^y))$.

In the hypo-elliptic case we can calculate the derivatives of $y \mapsto E(f(Y_t^y))$ with respect to the initial value by weight formulas, i.e. for any direction $v \in \mathbb{R}^N$ there is a random variable π such that

$$\frac{d}{d\epsilon}|_{\epsilon=0}E(f(Y_t^{y+\epsilon v}))) = E(f(Y_t^y)\pi) \tag{1.1}$$

for bounded measurable f. Explicit formulas for π can be given by Malliavin Calculus (see for instance [5] and [13]) and are in principle well-known since long time (see for instance [2]).

If we fix an order of approximation $m \geq 1$ and a (homogenous) direction v, such that (for the notations see Section 2)

$$v = \sum_{\substack{I\in\mathcal{A}\setminus\{\emptyset,(0)\}\\ \deg(I)=k}} v_I[\cdots[V_{i_1},V_{i_2}],V_{i_3}],\ldots,V_{i_k}](y)$$

holds with some $1 \leq k \leq m$, then we can construct a universal weight $\pi_{d,1}^m$ (calculated in a *universal way* from v_I and the Brownian motions, see Remark 8), such that

$$\frac{d}{d\epsilon}|_{\epsilon=0}E(f(Y_t^{y+\epsilon v})))t^{\frac{k}{2}} = E(f(Y_t^y)\pi_{d,1}^m) + \mathcal{O}(t^{\frac{m+1}{2}})$$

holds true, where the constant in the order estimate only depends on the first derivative of f. Theorem 8 asserts that we are able to find weights $\mu_j \neq 0$ and H^1-trajectories $\omega_j : [0,T] \to \mathbb{R}^{d+1}$ such that

$$\frac{d}{d\epsilon}|_{\epsilon=0}E(f(Y_t^{y+\epsilon v}))t^{\frac{k}{2}} = \sum_{j=1}^{r}\mu_j f(Y_t^y(\omega_j)) + \mathcal{O}(t^{\frac{m+1}{2}}) \tag{1.2}$$

holds true. Also here the constant in the order estimate depends on derivatives of f up to order $m+1$. The number r can be estimated similarly as in [8].

A combination of the original Cubature formula and formula (1.2) then yields by iteration a high order numerical approximation scheme, i.e.

$$\frac{d}{d\epsilon}|_{\epsilon=0}E(f(Y_t^{y+\epsilon v})) = \frac{d}{d\epsilon}|_{\epsilon=0}E(E(f(Y_t^{y+\epsilon v})|\mathcal{F}_{s_0})) \tag{1.3}$$

$$= \frac{d}{d\epsilon}|_{\epsilon=0}E(E(f(Y_{t-s_0}^z))|_{z=Y_{s_0}^{y+\epsilon v}}). \tag{1.4}$$

Calculation of the inner expectation by the original cubature formula and of the outer expectation by formula (1.2) lead to the desired iterative procedure: in the

uniformly hypo-elliptic case (see [8] for details) we consequently obtain the following recipe.

- choose a small $s_0 > 0$ and relatively large $t - s_0$. Subdivide $s = t_0 < t_1 < \cdots < t_k = t$ with $s_i := t_i - t_{i-1}$ for $i = 1, \ldots, k$.
- calculate by (1.2) with the function $z \mapsto E(f(Y^z_{t-s_0}))$ for a certain degree of accuracy m. The order estimate yields (by careful choice of homogenous) directions $\mathcal{O}(s^{\frac{m+1-k}{2}})$ for some $1 \le k \le m$. The constant in the order estimate depends on the $(m+1)$st derivative of $z \mapsto E(f(Y^z_{t-s}))$, which – by methods of Malliavin Calculus – can be reduced to dependence on the first derivative of f, i.e. there is an absolute bound given by
$$K||f||_{Lip} s_0^{\frac{m+1-k}{2}},$$
since $t - s_0$ is bounded away from 0.
- calculate $z \mapsto E(f(Y^z_{t-s}))$ by Cubature formulas with certain degree of accuracy m' (see [8]). The order estimate, which again – by methods from Malliavin Calculus (see [6]) – can be reduced to dependence on the first derivative of f, i.e. there is an absolute bound given by
$$K||f||_{Lip} \left(\sum_{i=1}^{k-1} \frac{s_i^{\frac{m'+1}{2}}}{(t-t_i)^{\frac{m'}{2}}} + s_k^{\frac{1}{2}} \right).$$
- finally we obtain a high order approximation scheme for $\frac{d}{d\epsilon}|_{\epsilon=0} E(f(Y_t^{y+\epsilon v}))$ through solving ordinary differential equations with estimate of the absolute error through
$$K||f||_{Lip} \left(s_0^{\frac{m+1-k}{2}} + \sum_{i=1}^{k-1} \frac{s_i^{\frac{m'+1}{2}}}{(t-t_i)^{\frac{m'}{2}}} + s_k^{\frac{1}{2}} \right).$$

  The procedure only involves ordinary differential equations and knowledge on the cubature trajectories. By choosing m big enough, one can bound the difference $m - k$ from below.

Remark 1. *A direct application of formula (1.1) is often very slow numerically in the purely hypo-elliptic case (see* [5]*). In the elliptic case there are very efficient ways to simulate π (see* [4]*). Hence in particular the purely hypo-elliptic case constitutes an interesting field of applications, since one does not need to worry about invertibility of the covariance matrix.*

Remark 2. *For all cubature formulas there are algebraic methods to determine the trajectories ω_j and the weights λ_j, or μ_j, respectively. These methods are based on calculus in nilpotent Lie groups, which constitute* the *local model for hypo-elliptic diffusions in the sense of Gromov (see for instance the very well written monograph* [11] *for a general outline, and* [1], [7] *for applications in the theory of – stochastic – differential equations).*

2. Cubature on Wiener Space

Cubature formulas on Wiener space (see the seminal article [8]) rely on the analysis of hypo-elliptic diffusions on free, nilpotent groups on the one hand (see the recent book [1]), on the other hand on stochastic Taylor expansion (see for instance [3]). Furthermore the proof of a Cubature formula on Wiener space appears as an

application of Tchakaloff's Theorem on finite dimensional cubature formulas (see [12]). We state these three main ingredients and show – in order to explain the mathematical background of Cubature formulas – a sketch of the proof (following [8]).

We fix a probability space $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ together with a d-dimensional Brownian motion $(B^i_s)_{0\leq t\leq T, i=1,\dots,d}$. We need the following notations for convenience:

(1) We abbreviate by $\mathcal{A}$ the set of all finite sequences $I := (i_1,\dots,i_k) \in \{0,\dots,d\}^k$, $k\geq 0$ and we define a degree function on $\mathcal{A}$ by

$$\deg(i_1,\dots,i_k) := k + \#\{l \text{ with } i_l = 0\},$$

which simply means that appearing 0s are counted twice. Additionally, we define $\deg(\emptyset)=0$. We define a semi-group structure on $\mathcal{A}$ via

$$\begin{aligned}(i_1,\dots,i_k) * (j_1,\dots,j_l) &:= (i_1,\dots,i_k,j_1,\dots,j_l),\\ \emptyset * (i_1,\dots,i_k) = (i_1,\dots,i_k) * \emptyset &= (i_1,\dots,i_k).\end{aligned}$$

(2) We denote by $\mathbb{A}^m_{d,1}$ the free, nilpotent algebra with $d+1$ generators $e_0,\dots,e_d$, i.e. the set of all non-commutative polynomials in those variables, such that the following nilpotency relations hold: if $\deg(I) > m$, then $e_{i_1}e_{i_2}\cdots e_{i_k} = 0$ for all $I\in\mathcal{A}$. Hence $\mathbb{A}^m_{d,1}$ is a finite dimensional, non-commutative, real algebra with unit element 1 and we are given a grading via the degree functions, i.e. a monomial $e_{i_1}e_{i_2}\cdots e_{i_k}$ is said to have degree n if $\deg(i_1,\dots,i_k) = n$. Denote by W_n the linear span of all monomials of degree n, then we obtain

$$\mathbb{A}^m_{d,1} = \oplus^m_{n=0} W_n,$$

furthermore $W_pW_q \subset W_{p+q}$ (where we define $W_p = 0$ for $p > m$ due to the given relations), $W_0 = \mathbb{R}\cdot 1$, so $\mathbb{A}^m_{d,1}$ is a graded algebra. We denote the canonical projections of x on the subspaces W_n by x_n for $n\geq 0$.

(3) On the finite dimensional algebra $\mathbb{A}^m_{d,1}$ we define the exponential series

$$\exp(x) := \sum_{i=0}^{\infty}\frac{x^i}{i!},$$

where the series converges everywhere due to the nilpotency relations. We define the logarithm on elements x with $x_0\neq 0$. We identify x_0 with a real number, so the series

$$\log(x) = \log(x_0) + \sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i}\left(\frac{x-x_0}{x_0}\right)^i.$$

is well-defined, since $\frac{x-x_0}{x_0}\in\oplus^m_{n=1}W_n$ (i.e. the series is finite).

(4) We define the Lie algebra $\mathfrak{g}^m_{d,1}$ generated by $e_0,\dots,e_d$ with respect to the Lie bracket $[x,y] := xy - yx$. The Lie algebra inherits the grading from the algebra $\mathbb{A}^m_{d,1}$ via $U_n := \mathfrak{g}^m_{d,1}\cap W_n$, for $n\geq 0$. Hence

$$\mathfrak{g}^m_{d,1} = \oplus^m_{n=1}U_n$$

is a graded Lie algebra. In fact the Lie algebra is free, nilpotent of step m, with d generators of degree 1 and one generator of degree 2.

(5) We denote the exponential image of $\mathfrak{g}^m_{d,1}$ by $G^m_{d,1} := \exp(\mathfrak{g}^m_{d,1}) \subset \mathbb{A}^m_{d,1}$ and call it the free, nilpotent Lie group. $G^m_{d,1}$ is indeed a Lie group as closed subgroup of Lie group $1 + \oplus_{n=1}^m W_n$. The tangent space at $x \in G^m_d$ is spanned by the left (or right) translations xw for $w \in \mathfrak{g}^m_{d,1}$.

(6) We need the canonical dilatations on Lie algebra and Lie group. We define for $x \in \mathbb{A}^m_{d,1}$ an algebra homomorphism Δ_t via

$$\Delta_t(x_n) = t^n x_n$$

for $x_n \in W_n$, $n \geq 1$ and $t > 0$. The homomorphism is well defined and restricts to a Lie algebra homomorphism Δ_t on $\mathfrak{g}^m_{d,1}$ and and a Lie group homomorphism Δ_t for $t > 0$.

(7) We apply the notation $\circ dB^0_t := dt$. We abbreviate the left invariant vector field associated to e_i by D_i, i.e. $D_i(x) = xe_i$ for $x \in G^m_{d,1}$. We define a stochastic process $(X^x_t)_{0\leq t\leq T}$ on $G^m_{d,1}$ via

$$dX^x_t = \sum_{i=0}^d D_i(X^x_t) \circ dB^i_t \tag{2.1}$$

$$= \sum_{i=0}^d X^x_t e_i \circ dB^i_t, \tag{2.2}$$

$$X^x_0 = x, \tag{2.3}$$

and see immediately that $X^x_t = xX^1_t$ for $0 \leq t \leq T$ almost surely.

(8) Note here and in the sequel that vector fields on a vector space $V : \mathbb{R}^N \to \mathbb{R}^N$ are used with double meaning: either as tangent directions on a smooth geometric object, or as first order differential operators on smooth functions $f : \mathbb{R}^N \to \mathbb{R}$,

$$(Vf)(y) := df(x) \cdot V(x)$$

for $x \in \mathbb{R}^N$. For smooth maps $G : \mathbb{R}^N \to \mathbb{R}^M$ we apply the notion of the *tangent map or Jacobian*

$$dG(x) \cdot v := \frac{d}{d\epsilon}|_{\epsilon=0} G(x + \epsilon v) \tag{2.4}$$

for $x, v \in \mathbb{R}^N$. A vector field is called C^∞*-bounded* if all derivatives of order greater of order greater than 0 are bounded.

In order to see the relation between free, nilpotent Lie groups on the one hand and asymptotic analysis as $t \downarrow 0$ on the other hand, we formulate stochastic Taylor expansion (see [1] and [3]), which simply results from iterating the defining equations for a stochastic differential equation.

Theorem 1 (stochastic Taylor expansion)**.** *Given* C^∞*-bounded vector fields*

$$V_0, \ldots, V_d : \mathbb{R}^N \to \mathbb{R}^N$$

on $\mathbb{R}^N$*, then the diffusion process*

$$dY^y_t = \sum_{i=0}^d V_i(Y^y_t) \circ dB^i_t,$$

$$Y^y_0 = y,$$

for $y \in \mathbb{R}^N$ admits the following series expansion: for $n \geq 0$ and $f \in C_b^\infty(\mathbb{R}^N)$ the expansion

$$\begin{aligned} f(Y_t^y) = & \sum_{\substack{I \in \mathcal{A} \\ \deg(I) \leq n}} V_{i_1} \cdots V_{i_k} f(y) \int_{0 \leq t_1 \leq \cdots \leq t_k \leq t} \circ dB_{t_1}^{i_1} \circ \cdots \circ dB_{t_k}^{i_k} + \\ & + R_n(t,f), \\ R_n(t,f) = & \sum_{\substack{(i_1,\ldots,i_k) \in \mathcal{A} \\ \deg(i_1,\ldots,i_k) \leq n \\ (i_0,\ldots,i_k) \in \mathcal{A} \\ \deg(i_0,\ldots,i_k) > n}} \int_{0 \leq t_0 \leq \cdots \leq t_k \leq t} V_{i_0} \cdots V_{i_k} f(Y_{t_0}^y) \circ dB_{t_0}^{i_0} \circ \cdots \circ dB_{t_k}^{i_k}, \end{aligned}$$

holds true for $0 \leq t \leq T$. Additionally $\sqrt{E(R_n(t,f)^2)} = \mathcal{O}(t^{\frac{m+1}{2}})$, where the constant in the order estimate depends on the $(m+1)$st derivative of f.

Remark 3. *The above procedure works for "any" differential equation (see the literature on rough paths, for instance the seminal work [7]), in particular, if we consider the non-autonomous, ordinary differential equation*

$$dZ_t^y = \sum_{i=0}^d V_i(Z_t^y) d\omega^i(t)$$

for a H^1-trajectory $\omega : [0,T] \to \mathbb{R}^{d+1}$, we have for $n \geq 0$ and $f \in C_b^\infty(\mathbb{R}^N)$ the expansion

$$\begin{aligned} f(Z_t^y) = & \sum_{\substack{I \in \mathcal{A} \\ \deg(I) \leq n}} V_{i_1} \cdots V_{i_k} f(y) \int_{0 \leq t_1 \leq \cdots \leq t_k \leq t} d\omega^{i_1}(t_1) \cdots d\omega^{i_k}(t_k) + \\ & + R_n(t,f), \\ R_n(t,f) = & \sum_{\substack{(i_1,\ldots,i_k) \in \mathcal{A} \\ \deg(i_1,\ldots,i_k) \leq n \\ (i_0,\ldots,i_k) \in \mathcal{A} \\ \deg(i_0,\ldots,i_k) > n}} \int_{0 \leq t_0 \leq \cdots \leq t_k \leq t} V_{i_0} \cdots V_{i_k} f(Z_{t_0}^y) d\omega^{i_0}(t_0) \cdots d\omega^{i_k}(t_k), \end{aligned}$$

holds true for $0 \leq t \leq T$. Notice here that in order to obtain an asymptotic expansion (for a given fixed trajectory ω) of certain order in time t, one has to change the degree function.

Example 1. *If we apply this series expansion to the process X_t^x in the vector space $\mathbb{A}_{d,1}^m$, we obtain for any linear function $\lambda : \mathbb{A}_{d,1}^m \to \mathbb{R}$*

$$\lambda(X_t^x) = \sum_{\substack{I \in \mathcal{A} \\ \deg(I) \leq m}} \lambda(x e_{i_1} \cdots e_{i_k}) \int_{0 \leq t_1 \leq \cdots \leq t_k \leq t} \circ dB_{t_1}^{i_1} \circ \cdots \circ dB_{t_k}^{i_k}$$

for $x \in \mathbb{A}_{d,1}^m$ and $0 \leq t \leq T$, since

$$D_i \lambda(x) := \frac{d}{d\epsilon}|_{\epsilon=0} \lambda(x + \epsilon x e_i) = \lambda(x e_i)$$

for $e_i \in \mathfrak{g}_{d,1}^m$, hence

$$(D_{i_1} \cdots D_{i_k} \lambda)(x) = \lambda(x e_{i_1} \cdots e_{i_k}).$$

Consequently we obtain in $\mathbb{A}^m_{d,1}$

$$X^1_t = \sum_{\substack{I\in\mathcal{A}\\ \deg(I)\le m}} e_{i_1}\cdots e_{i_k}\int_{0\le t_1\le\cdots\le t_k\le t}\circ dB^{i_1}_{t_1}\circ\cdots\circ dB^{i_k}_{t_k}$$

for $0\le t\le T$*, almost surely, by duality. This formula provides a nice stochastic representation of the solution of equation 2.1, where we apply that* $G^m_{d,1}$ *is embedded in the algebra* $\mathbb{A}^m_{d,1}$*.*

Stochastic differential equations provide solutions for the associated heat equation via expectations, so we obtain for $f\in C^\infty_b(\mathbb{A}^m_{d,1})$,

$$(D_0+\frac{1}{2}\sum_{i=1}^d D_i^2)(x\mapsto E(f(X^x_t))) = \frac{\partial}{\partial t}E(f(X^x_t)),$$

for all $x\in\mathbb{A}^m_{d,1}$. The heat equation itself admits a similar construction as stochastic Taylor expansion, namely any solution $u(t,x)$ with initial value $u(0,x)=f(x)$, where f is C^∞-bounded, can be written as

$$u(t,x)=\sum_{j=0}^n\frac{t^j}{j!}(D_0+\frac{1}{2}\sum_{i=1}^d D_i^2)^j f(x)+R_n(t,f),$$

$$R_n(t,f)=\int_0^t\cdots\int_0^{s_1}(D_0+\frac{1}{2}\sum_{i=1}^d D_i^2)^{n+1}u(s_0,x)ds_0\dots ds_n.$$

In the case of $f=\lambda$ for a linear function $\lambda:\mathbb{A}^m_{d,1}\to\mathbb{R}$, this leads for the process X^x_t in $\mathbb{A}^m_{d,1}$, to the nice formulas

$$E(\lambda(X^x_t)) = \sum_{\substack{I\in\mathcal{A}\\ \deg(I)\le m}}\lambda(xe_{i_1}\cdots e_{i_k})E(\int_{0\le t_1\le\cdots\le t_k\le t}\circ dB^{i_1}_{t_1}\circ\cdots\circ dB^{i_k}_{t_k}),$$

$$u(t,x)=\lambda(x\exp(t(e_0+\frac{1}{2}\sum_{i=1}^d e_i^2))),$$

and hence

$$\exp(t(e_0+\frac{1}{2}\sum_{i=1}^d e_i^2)) = \sum_{\substack{I\in\mathcal{A}\\ \deg(I)\le m}} e_{i_1}\cdots e_{i_k}E(\int_{0\le t_1\le\cdots\le t_k\le t}\circ dB^{i_1}_{t_1}\circ\cdots\circ dB^{i_k}_{t_k}), \tag{2.5}$$

for $0\le t\le T$, which is one basic formula for the construction of Cubature formulas (see [1] and [8]).

In order to obtain a cubature formula we need a version of Tchakaloff's theorem (see [12] for details in the case where the support of μ is not compact):

Theorem 2. *Let* μ *be a measure on* $\mathbb{R}^N$ *such that moments of all orders exist. Fix a number* m*, then there are points* $x_1,\dots,x_r\in\operatorname{supp}(\mu)$ *and weights* $\lambda_j>0$ *for* $j=1,\dots,r$*, such that*

$$\int_{\mathbb{R}^N}f(y)\mu(dy)=\sum_{j=1}^r f(x_j)\lambda_j$$

for all polynomials on $\mathbb{R}^N$ *up to order* m*. Furthermore* $r\le\dim_{\mathbb{R}}\operatorname{Pol}_m(\mathbb{R}^N)$*, where* $\operatorname{Pol}_m(\mathbb{R}^N)$ *is the vector space of polynomials on* $\mathbb{R}^N$ *with degree less or equal* m*.*

On basis of these prerequisites we can define cubature formulas on Wiener space:

Definition 1. *Fix $m \geq 1$ and $0 < t \leq T$, a cubature formula on Wiener space is given by a finite number of points $x_1, \dots, x_r \in G^m_{d,1}$ and finitely many weights $\lambda_1, \dots, \lambda_r > 0$, such that*

$$E(X^1_t) = \sum_{j=1}^{r} \lambda_j x_j,$$

or equivalently due to formula (2.5)

$$\exp(t(e_0 + \frac{1}{2}\sum_{i=1}^{d} e_i^2)) = \sum_{j=1}^{r} \lambda_j x_j.$$

Furthermore $r \leq \dim_{\mathbb{R}} \mathbb{A}^m_{d,1}$.

Applying Tchakaloff's theorem to the law of the process X^1_t in $\mathbb{A}^m_{d,1}$, $0 < t \leq T$, which is supported in $G^m_{d,1}$, yields that we find points $x_1, \dots, x_r \in \operatorname{supp}((X^1_t)_* P) \subset G^m_{d,1}$ and weights $\lambda_j > 0$, such that

$$E(X^1_t) = \sum_{\substack{I \in \mathcal{A} \\ \deg(I) \leq m}} e_{i_1} \cdots e_{i_k} E(\int_{0 \leq t_1 \leq \cdots \leq t_k \leq t} \circ dB^{i_1}_{t_1} \circ \cdots \circ dB^{i_k}_{t_k}) = \sum_{j=1}^{r} \lambda_j x_j,$$

whence the existence of Cubature formulas for any $m \geq 1$.

Finally we apply Chow's theorem of sub-riemannian geometry (see [1] and [11]), which tells that every point $x \in G^m_{d,1}$ can be reached by a H^1-horizontal curve, i.e. for every $x \in G^m_{d,1}$ we find a H^1-curve $\omega : [0,T] \to \mathbb{R}^{d+1}$, such that the solution of the non-autonomous ordinary differential equation

$$dZ^1_t(\omega) = \sum_{i=0}^{d} D_i(Z^1_t(\omega)) d\omega^i(t)$$

reaches x at time $t > 0$, i.e. $Z^1_t(\omega) = x$.

Taking for each of the x_j appropriate curves ω_j with the previous property, we obtain – by Taylor expansion of the solution of the non-autonomuous equation as in Remark 3 – the familiar version of cubature formulas on Wiener space, namely

$$\sum_{\substack{I \in \mathcal{A} \\ \deg(I) \leq m}} e_{i_1} \cdots e_{i_k} E(\int_{0 \leq t_1 \leq \cdots \leq t_k \leq t} \circ dB^{i_1}_{t_1} \circ \cdots \circ dB^{i_k}_{t_k}) =$$

$$\sum_{j=1}^{r} \lambda_j \sum_{\substack{I \in \mathcal{A} \\ \deg(I) \leq m}} e_{i_1} \cdots e_{i_k} \int_{0 \leq t_1 \leq \cdots \leq t_k \leq t} d\omega^{i_1}_j(t_1) \cdots d\omega^{i_k}_j(t_k).$$

This leads to a redefinition of cubature formulas, where we replace the points x_i by endpoints of evolutions of ordinary differential equations $Z^1_t(\omega_i) = x_i$ for $i = 1, \dots, r$.

Remark 4. *As proposed in [8] we could also find for any point $x \in G^m_{d,1}$ a H^1-trajectory $\widetilde{\omega} : [0,T] \to \mathbb{R}^d$, such that $\omega(t) := (t, \widetilde{\omega}(t))$ for $t \in [0,T]$ satisfies $Z^1_t(\omega) = x$, hence we could equally work with this simpler class of curves, which we do not do in this article.*

Theorem 3. *Given C^∞-bounded vector fields $V_0,\dots,V_d$ on $\mathbb{R}^N$, then the diffusion process*

$$dY_t^y = \sum_{i=0}^d V_i(Y_t^y)\circ dB_t^i,$$
$$Y_0^y = y,$$

for $y\in\mathbb{R}^N$ admits the following cubature formula of degree $m\geq 1$: for $0<t\leq T$ we find H^1-curves $\omega_1,\dots,\omega_r:[0,T]\to\mathbb{R}^{d+1}$ and weights $\lambda_1,\dots,\lambda_r>0$, such that

$$E(f(Y_t^y)) = \sum_{j=1}^r \lambda_j f(Y_t^y(\omega_j)) + \mathcal{O}(t^{\frac{m+1}{2}}),$$

for $f\in C_b^\infty(\mathbb{R}^N)$. The constant for the order estimate depends in general on derivatives of f up to order $m+1$. The curve $(Y_t^y(\omega))_{0\leq t\leq T}$ is understood as solution of

$$dY_t^y(\omega) = \sum_{i=0}^d V_i(Y_t^y(\omega))d\omega^i(t),$$
$$Y_0^y(\omega) = y.$$

Remark 5. *Notice that in the hypo-elliptic case we can improve Theorem 3 by methods proved in* [6]*. Fix $m\geq 1$. A family of random variables $\{Z_I$ for $I\in\mathcal{A},\deg(I)\leq m\}$ is called m-moment generating if $Z_\emptyset=1$ and*

$$E(Z_{I_1}Z_{I_2}\cdots Z_{I_l}) = E(B^{\circ I_1}(1)\cdots B^{\circ I_l}(1))$$

for all $I_j\in\mathcal{A}$ with $\deg(I_1)+\cdots+\deg(I_l)\leq m$. Here we apply

$$B^{\circ\emptyset}(t) = 1$$
$$B^{\circ I*(i)}(t) := \int_0^t B^{\circ I}(s)\circ dB_s^i$$

for $0\leq t\leq T$. The results on order estimates in [6] *apply for m-moment-generating families.*

Given now a cubature formula, the coefficients $\int_{0\leq t_1\leq\cdots\leq t_k\leq 1} d\omega_j^{i_1}(t_1)\cdots d\omega_j^{i_k}(t_k)$ in the basis $e_{i_1}\cdots e_{i_k}$, which appear with probability λ_j constitute a m-moment generating family (see [8]*): this is basically due to the fact that products of iterated integrals like $B^{\circ I_1}(1)\cdots B^{\circ I_l}(1)$ can be decomposed as linear combination of iterated integrals (for instance $B_1^1B_1^2 = B^{\circ(1,2)}(1)+B^{\circ(2,1)}(1)$), hence the assertion on expectations of products follows from the hypothesis on the expectation of iterated integrals. Equivalently one can see this phenomenon also by transforming the defining differential equation for the process $(X_t^x)_{t\geq 0}$ to the Lie algebra in the exponential chart. It is remarkable that – due to this observation – we obtain*

$$E((X_1^1)^k) = \sum_{i=1}^r \lambda_i (Z_1^1(\omega_i))^k,$$

for all $0\leq k\leq m$, only if the equation holds for $k=1$, hence

$$E(\log(X_1^1)) = \sum_{i=1}^r \lambda_i \log Z_1^1(\omega_i)$$

by definition of the logarithm.

3. Calculation of the Greeks

For the calculation of Greeks we proceed by methods from Malliavin Calculus (see [10] and [9] for all details). We shall consider stochastic differential equations of the type

$$dY_t^y = \sum_{i=0}^{d} V_i(Y_t^y) \circ dB_t^i,$$
$$Y_0^x = y,$$

on a stochastic basis $(\Omega, \mathcal{F}, P)$, where we are given a d-dimensional Brownian motion $(B_t^i)_{0 \leq t \leq T, i=1,\dots,d}$ in its natural filtration, up to a finite time horizon $T > 0$. For the vector fields V_i we shall assume the following regularity assertion:

- The vector fields
$$V_i : \mathbb{R}^N \to \mathbb{R}^N$$
are C^∞-bounded.

If the distribution

$$\mathcal{D}_{LA}(y) := \langle V_1(y), \dots, V_d(y), [V_i, V_j](y), \dots \text{ for } i, j, \dots = 0, \dots, d \rangle \tag{3.1}$$

has constant rank N for $y \in \mathbb{R}^N$, we say that *Hörmander's condition* holds. Here and in the sequel we apply the notion of Lie brackets of two vector fields $V, W : \mathbb{R}^N \to \mathbb{R}^N$, $[V, W](y) := dV(y) \cdot W(y) - dW(y) \cdot V(y)$ for $y \in \mathbb{R}^N$.

Theorem 4. *Fix $t > 0$, $v \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$. If condition (3.1) holds, there is a weight $\pi \in \mathcal{D}^\infty$, such that for all bounded, measurable functions $f : \mathbb{R}^N \to \mathbb{R}$ the equation*

$$\frac{d}{d\epsilon}|_{\epsilon=0} E(f(Y_t^{y+\epsilon v})) = E(f(Y_t^y)\pi)$$

holds true depending on t, v, y and the whole stochastic process $(Y_t^y)_{0 \leq t \leq T}$.

Proof. For the proof see [5] and [13]. □

We shall apply usual notions of Malliavin calculus (see [9] and [10]). The first variation $J_{0\to t}(y)$ denotes the derivative of $(Y_t^y)_{0 \leq t \leq T}$ with respect to y, hence

$$dJ_{0\to t}(y) = \sum_{i=0}^{d} dV_i(Y_t^y) J_{0\to t}(y) \circ dB_t^i$$
$$J_{0\to 0}(y) = id_N,$$

where we apply the Jacobian dV_i of the vector field V_i as defined in 2.4. This is an almost surely invertible process and we obtain the representation

$$D_s^k Y_t^y = J_{0\to t}(y)(J_{0\to s}(y))^{-1} V_k(Y_s^y) 1_{[0,t]}(s)$$

for $0 \leq s \leq T$, $0 \leq t \leq T$, $1 \leq k \leq d$ of the Malliavin derivative. We have the fundamental partial integration formula

$$\sum_{k=1}^{d} E(\int_0^T D_s^k Y_t^y a^k ds) = E(Y_t^y \delta(a)),$$

where $(a_s)_{0\leq s\leq T} := (a^k_{0\leq s\leq T})_{k=1,\dots,d} \in \mathrm{dom}(\delta)$ is a Skorohod integrable process. $\delta(a)$ is real valued random variable. Notice that if a is predictable and square-inegrable, then

$$\delta(a) = \sum_{k=1}^{d} \int_0^T a^k_s dB^k_s,$$

hence the Skorohod integral coincides with the Ito integral.

For the pull-back of the stochastic flow, we can provide a similar stochastic Taylor expansion on the space of vector fields as in Section 2, or – in the case of a Lie group – on the Lie algebra. We apply the following notation:

$$\begin{aligned}\mathcal{L}_\emptyset V &:= V\\ \mathcal{L}_i V &:= [V, V_i],\\ \mathcal{L}_{(I)} V &:= \mathcal{L}_{i_1} \circ \cdots \circ \mathcal{L}_{i_k}(V),\end{aligned}$$

for $I = (i_1, \dots, i_k) \in \mathcal{A}$.

Theorem 5. *Given C^∞-bounded vector fields $V_0, \dots, V_d$ on $\mathbb{R}^N$, then the diffusion process*

$$\begin{aligned}dY^y_t &= \sum_{i=0}^{d} V_i(Y^y_t) \circ dB^i_t,\\ Y^y_0 &= y,\end{aligned}$$

for $y \in \mathbb{R}^N$, admits the following series expansion for the pull back $J_{0\to t}(y)^{-1}V(Y^y_t)$ on smooth vector fields $V : \mathbb{R}^N \to \mathbb{R}^N$.

$$J_{0\to t}(y)^{-1}V(Y^y_t) = \sum_{\substack{I\in\mathcal{A}\\ \deg(I)\leq n}} \mathcal{L}_{(I)}V(y) \int_{0\leq t_1\leq\cdots\leq t_k\leq t} \circ dB^{i_1}_{t_1} \circ \cdots \circ dB^{i_k}_{t_k} + R_n(t, V),$$

for $0 \leq t \leq T$. For the remainder term we obtain

$$\begin{aligned}&R_n(t, V)\\ &= \sum_{\substack{(i_1,\dots,i_k)\in\mathcal{A}\\ \deg(i_1,\dots,i_k)\leq n\\ (i_0,\dots,i_k)\in\mathcal{A}\\ \deg(i_0,\dots,i_k)>n}} \int_{0\leq t_0\leq\cdots\leq t_k\leq t} J_{0\to t_0}(y)^{-1}\mathcal{L}_{(i_0,\dots,i_k)}V(Y^y_{t_0}) \circ dB^{i_0}_{t_0} \circ \cdots \circ dB^{i_k}_{t_k},\end{aligned}$$

for $0 \leq t \leq T$.

This Taylor expansion leads on the Lie group $G^m_{d,1}$ to an action on the Lie algebra $\mathfrak{g}^m_{d,1}$, namely

$$X^1_t w (X^1_t)^{-1} = \sum_{\substack{I\in\mathcal{A}\\ \deg(I)\leq m-1}} [e_{i_1}, \cdots [e_{i_k}, w] \cdots] \int_{0\leq t_1\leq\cdots\leq t_k\leq t} \circ dB^{i_1}_{t_1} \circ \cdots \circ dB^{i_k}_{t_k}$$

by inserting the vector fields D_i for V_i and evaluating at $x = 1$. Notice the decrease in order from m to $m-1$, since in $\mathfrak{g}^m_{d,1}$ Lie brackets only up to order $m-1$ can be non-zero. This stochastic process on the Lie algebra $\mathfrak{g}^m_{d,1}$ will be applied in the sequel in order to construct approximate, universal weights for the calculation of the Greeks. First we need a Lemma on scaling invariance.

Lemma 1. *The following identities in law for the stochastic process $(X_s^x)_{0\le s\le T}$ hold,*

$$(\Delta_{\sqrt{t}}X_s^x)_{0\le s\le 1} =_{law} (X_{st}^x)_{0\le s\le 1},$$
$$(\Delta_{\sqrt{t}}[(X_s^e)w(X_s^e)^{-1}])_{0\le s\le 1} =_{law} ((\sqrt{t})^{\deg(w)}(X_{st}^e)e_i(X_{st}^e)^{-1})_{0\le s\le 1}$$

for $t>0$ and $w\in W_n$ (so $\deg(w)=n$).

Proof. By scaling invariance we know that $(\sqrt{t}B_s^i)_{0\le s\le 1} =_{\text{law}} (B_{st}^i)_{0\le s\le 1}$ for $t>0$ and $i=1,\dots,d$. This extends to all iterated stochastic integrals. Notice the importance of the degree function, which associates degree 2 to e_0, and hence the correct scaling property. Notice furthermore that for $w\in\mathfrak{g}_{d,1}^m\cap W_n$ the element $(X_s^e)w(X_s^e)^{-1}$ is an element of the Lie algebra $\mathfrak{g}_{d,1}^m$, but not necessarily of W_n! □

Remark 6. *Notice that the stochastic process $(X_t^x)_{0\le t\le T}$ is not hypo-elliptic, since the direction e_0 always points into directions where the density does not admit a logarithmic derivative. Therefore we only calculate with directions $w\in\mathfrak{g}_{d,1}^m/\langle e_0\rangle$, which means directions with vanishing e_0 component. In all those directions we are able to conclude a result on differentiability.*

Remark 7. *In the sequel we shall construct Skorohod integrable processes $(a_s^i)_{0\le s\le T}$ by defining them only on $[0,t]$ for $0<t\le T$. It is understood that these processes are defined on $[0,T]$, but vanish for $s>t$, hence also $a_s^i=a_s^i1_{[0,T]}(s)$ for $0\le s\le T$ and $i=1,\dots,d$.*

Proposition 1. *Fix $w\in\mathfrak{g}_{d,1}^m/\langle e_0\rangle$ (no e_0 component). For $t>0$ there are Skorohod-integrable processes $(a_s^i)_{0\le s\le T}$ (the t-dependence is not visible in our notation) with $a_s^i=a_s^i1_{[0,T]}(s)$ for $0\le s\le T$, such that*

$$\sum_{i=1}^d\int_0^t(X_s^1)e_i(X_s^1)^{-1}a_s^i ds=\Delta_{\sqrt{t}}(w)$$

and

$$\sum_{i=1}^d E(\int_0^t(a_s^i)^2ds)=\mathcal{O}(1).$$

Furthermore we obtain that for any bounded measurable function $f:G_{d,1}^m\to\mathbb{R}$, the equation

$$\frac{d}{d\epsilon}|_{\epsilon=0}E(f(X_t^{x+\epsilon\Delta_{\sqrt{t}}(w)}))=E(f(X_t^x)\pi_{d,1}^m)$$

holds true, where $\pi_{d,1}^m:=\delta(s\mapsto a_s)$ denotes the Skorohod integral of the strategies a.

Proof. The stochastic process $(X_t^x)_{0\le t\le T}$ is not hypo-elliptic, but we prove that derivatives in all directions $w\in\mathfrak{g}_{d,1}^m$ with vanishing e_0 component exist. We can define the processes a_s^i by the standard method of Malliavin Calculus: we choose a scalar product $\langle,\rangle$ on $\mathfrak{g}_{d,1}^m$, which respects the grading, i.e. W_k is orthogonal to W_l for $k\ne l$, and define the reduced covariance operator C^t as symmetric operator $C^t:\mathfrak{g}_{d,1}^m\to\mathfrak{g}_{d,1}^m$by

$$\langle y,C^ty\rangle:=\sum_{i=1}^d\int_0^t\left\langle(X_s^1)e_i(X_s^1)^{-1},y\right\rangle^2ds.$$

This operator is almost surely invertible on $\mathfrak{g}^m_{d,1}/\langle e_0\rangle$ with p-integrable inverse (see [10]). Obviously the image of C^t lies in $\mathfrak{g}^m_{d,1}/\langle e_0\rangle$, since there is no e_0-component in $(X^1_s)^{-1}e_iX^1_s$, the kernel of C^t can be calculated by the classical method from [10] and vanishes on $\mathfrak{g}^m_{d,1}/\langle e_0\rangle$, hence $C^t|_{\mathfrak{g}^m_{d,1}/\langle e_0\rangle}$ is invertible with p-integrable inverse, since the Norris-Lemma applies.

Furthermore we observe the following scaling property: notice that $\Delta_{\sqrt{t}}$ is a self-adjoint operator on $\mathfrak{g}^m_{d,1}$ for $t>0$. We can compare $\Delta_{\sqrt{t}}C^1\Delta_{\sqrt{t}}$ with C^t in law via Lemma 1

$$\begin{aligned}
\left\langle y, C^t y\right\rangle &= \sum_{i=1}^d \int_0^t \left\langle (X^1_s)e_i(X^1_s)^{-1}, y\right\rangle^2 ds \\
\left\langle \Delta_{\sqrt{t}}y, C^1\Delta_{\sqrt{t}}y\right\rangle &= \sum_{i=1}^d \int_0^1 \left\langle (X^1_s)e_i(X^1_s)^{-1}, \Delta_{\sqrt{t}}y\right\rangle^2 ds \\
&= \sum_{i=1}^d \int_0^1 \left\langle \Delta_{\sqrt{t}}[(X^1_s)e_i(X^1_s)^{-1}], y\right\rangle^2 ds \\
&=_{\text{law}} \sum_{i=1}^d \int_0^1 \left\langle \sqrt{t}(X^1_{st})e_i(X^1_{st})^{-1}, y\right\rangle^2 ds \\
&= \sum_{i=1}^d \int_0^t \left\langle (X^1_u)e_i(X^1_u)^{-1}, y\right\rangle^2 du.
\end{aligned}$$

So we have that

$$((X^1_s)e_i(X^1_s))_{0\le s\le t}, C^t) =_{\text{law}} ((\frac{1}{\sqrt{t}}\Delta_{\sqrt{t}}[(X^1_{\frac{s}{t}})e_i(X^1_{\frac{s}{t}})^{-1}])_{0\le s\le t}, \Delta_{\sqrt{t}}C^1\Delta_{\sqrt{t}})$$

for $t>0$. We define the (non-adapted) strategies via

$$a^i_s := \left\langle (X^1_s)e_i(X^1_s), (C^t|_{\mathfrak{g}^m_{d,1}/\langle e_0\rangle})^{-1}\Delta_{\sqrt{t}}(w)\right\rangle$$

for $0\le s\le t$, $i=1,\dots,d$, and obtain

$$\sum_{i=1}^d \int_0^t \left\langle (X^1_s)e_i(X^1_s)^{-1}, v\right\rangle a^i_s ds =$$
$$\sum_{i=1}^d \int_0^t \left\langle (X^1_s)e_i(X^1_s)^{-1}, v\right\rangle \left\langle (X^1_s)e_i(X^1_s)^{-1}, (C^t)^{-1}\Delta_{\sqrt{t}}(w)\right\rangle ds =$$
$$\left\langle v, C^t(C^t)^{-1}\Delta_{\sqrt{t}}(w)\right\rangle = \left\langle v, \Delta_{\sqrt{t}}(w)\right\rangle$$

for $v\in\mathfrak{g}^m_{d,1}$. Hence the strategies satisfy the desired equation by duality on $\mathfrak{g}^m_{d,1}$. Concerning the order estimate for the strategies we observe that

$$\begin{aligned}
&\left\langle (X^1_s)e_i(X^1_s)^{-1}, (C^t)^{-1}\Delta_{\sqrt{t}}(w)\right\rangle \\
&=_{\text{law}} \left\langle \frac{1}{\sqrt{t}}\Delta_{\sqrt{t}}[(X^1_{\frac{s}{t}})e_i(X^1_{\frac{s}{t}})^{-1}], \Delta^{-1}_{\sqrt{t}}(C^1)^{-1}\Delta^{-1}_{\sqrt{t}}\Delta_{\sqrt{t}}(w)\right\rangle \\
&= \left\langle \frac{1}{\sqrt{t}}(X^1_{\frac{s}{t}})e_i(X^1_{\frac{s}{t}})^{-1}, (C^1)^{-1}w\right\rangle,
\end{aligned}$$

so consequently

$$\sum_{i=1}^{d} E(\int_0^t (a_s^i)^2 ds) =_{\text{law}} \sum_{i=1}^{d} E(\int_0^t \left\langle \frac{1}{\sqrt{t}} (X^1_{\frac{s}{t}}) e_i (X^1_{\frac{s}{t}})^{-1}, (C^1)^{-1} w \right\rangle^2 ds$$
$$= \sum_{i=1}^{d} E(\int_0^t \left\langle \frac{1}{\sqrt{t}} (X^1_{\frac{s}{t}}) e_i (X^1_{\frac{s}{t}})^{-1}, (C^1)^{-1} w \right\rangle^2 td(\frac{s}{t})$$
$$= \sum_{i=1}^{d} E(\int_0^1 \left\langle (X^1_u) e_i (X^1_u)^{-1}, (C^1)^{-1} w \right\rangle^2 du$$

□

Remark 8. *The strategies are **universal** in the sense that they only depend on the Brownian motion, the choice of a scalar product on $\mathfrak{g}^m_{d,1}$, and – in a linear way – on the "input" direction w. We denote the Skorohod integrals of those strategies by $\pi^m_{d,1} := \delta(a)$ and call them* universal weights. *Notice also that $E((\pi^m_{d,1})^2) = \mathcal{O}(1)$, which is seen by a similar calculation as the previous one. Hence there is a universal time dependence, which is achieved by calculating the strategies with respect to the direction $\Delta_{\sqrt{t}} w$ instead of w. We omit the dependence on w in our notation.*

Example 2. *We consider the simplest non-trivial example: $m = 2$ and $d = 2$. Hence the nilpotent algebra is generated by*

$$W_0 = \langle 1 \rangle,$$
$$W_1 = \langle e_1, e_2 \rangle,$$
$$W_2 = \langle e_1 e_2, e_2 e_1, e_0 \rangle,$$

and the Lie algebra by $e_1, e_2, e_0, [e_1, e_2]$. The dimensions are 6 and 4 respectively. We can solve explicitly

$$dX_t^x = \sum_{i=0}^{2} X_t^x e_i \circ dB_t^i$$

by

$$X_t^x = x + xe_1 B_t^1 + xe_2 B_t^2 + xe_0 t + xe_1 e_2 \int_0^t \int_0^{t_2} \circ dB^1_{t_1} \circ dB^2_{t_2} +$$
$$+ xe_2 e_1 \int_0^t \int_0^{t_2} \circ dB^2_{t_1} \circ dB^1_{t_2} + \frac{1}{2} xe_1^2 (B_t^1)^2 + \frac{1}{2} xe_2^2 (B_t^2)^2.$$

The pull-back yields a shorter expression, namely

$$(X_t^1) e_1 (X_t^1)^{-1} = e_1 + [e_2, e_1] B_t^2,$$
$$(X_t^1) e_2 (X_t^1)^{-1} = e_2 + [e_1, e_2] B_t^1,$$

for $0 \leq t \leq T$, which can also be calculated directly. Consequently the above equation reduces to find a Skorohod integrable strategy such that

$$\int_0^t a_s^1 ds = \sqrt{t} w_1$$
$$\int_0^t a_s^2 ds = \sqrt{t} w_2$$
$$\int_0^t (a_s^1 B_s^2 - a_s^2 B_s^1) ds = t w_3$$

for $w = w_1 e_1 + w_2 e_2 + w_3 [e_1, e_2]$. This can be done by introducing a scalar product on $\mathfrak{g}_{d,1}^m$ such that $e_1, e_2, [e_1, e_2], e_0$ is an orthonormal basis. We then obtain the symmetric matrix

$$C^t = \begin{pmatrix} t & 0 & \int_0^t B_s^2 ds & 0 \\ 0 & t & -\int_0^t B_s^1 ds & 0 \\ \int_0^t B_s^2 ds & -\int_0^t B_s^1 ds & \int_0^t ((B_s^1)^2 + (B_s^2)^2) ds & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the given basis. So C^t is almost surely invertible on $\langle e_1, e_2, [e_1, e_2] \rangle$ – as claimed in Proposition 1, indeed the determinant is given through

$$\det(C^t|_{\langle e_1, e_2, [e_1, e_2] \rangle}) = t^2 \int_0^t ((B_s^1)^2 + (B_s^2)^2) ds - t (\int_0^t B_s^1 ds)^2 - t (\int_0^t B_s^2 ds)^2,$$

which is positive if and only if one of the two Brownian motions is not vanishing identically on $[0, t]$. The order assertion in t is also nicely visible.

We consider the first derivative of the function $y \mapsto E(f(Y_t^y))$ for $t > 0$. By Hörmander's theorem we know that this function is smooth for all bounded measurable functions $f : \mathbb{R}^N \to \mathbb{R}$ (see for instance [9] or [10]). We generalize this assertion on the one hand, since we leave away Hörmander's condition. On the other hand we only prove an approximative result with a certain time asymptotics.

Theorem 6. *Fix $t > 0$, $v \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$. Furthermore we fix an order $m \geq 1$ of approximation. We assume that the vector v is a linear combination of Lie brackets up to order $m - 1$ (except the direction V_0), i.e.*

$$v = \sum_{\substack{I \in \mathcal{A} \setminus \{\emptyset, (0)\} \\ \deg(I) \leq m-1}} t^{\frac{\deg(I)}{2}} w_I [V_{i_1}, [V_{i_2}, [\cdots, V_{i_k}] \cdots](y),$$

and define

$$\Delta_{\sqrt{t}}(w) := \sum_{\substack{I \in \mathcal{A} \setminus \{\emptyset, (0)\} \\ \deg(I) \leq m-1}} t^{\frac{\deg(I)}{2}} w_I [e_{i_1}, [e_{i_2}, [\cdots, e_{i_k}] \cdots].$$

Then there is universal weight $\pi = \pi_{d,1}^m \in \mathcal{D}^\infty$ associated to $\Delta_{\sqrt{t}}(w)$, such that for all C^∞-bounded functions $f : \mathbb{R}^N \to \mathbb{R}$ the equation

$$\frac{d}{d\epsilon}|_{\epsilon=0} E(f(Y_t^{y+\epsilon v})) = E(f(Y_t^y) \pi_{d,1}^m) + \mathcal{O}(t^{\frac{m+1}{2}})$$

holds true, where the constant in the order estimate only depends on the first derivative of f.

Proof. Assume f is C_b^1, then

$$\frac{d}{d\epsilon}|_{\epsilon=0}E(f(Y_t^{y+\epsilon v}))=E(df(Y_t^y)\cdot J_{0\to t}(y)\cdot v),$$

where df denotes the differential of the function f. Instead of solving the equation

$$v=\sum_{k=1}^d\int_0^t J_{0\to s}(y)^{-1}V_k(Y_s^y)a_s^k ds,$$

we take the universal weight $\pi_{d,1}^m=\delta(s\mapsto a_s)$ of Proposition 1 and solve the equation

$$v=\sum_{k=1}^d\int_0^t (J_{0\to s}(y)^{-1}V_k(Y_s^y)-R_m^k(s,V_k))a_s^k ds$$

by the above universal construction. More precisely, we choose real numbers w_I for $I\in\mathcal{A}\setminus\{\emptyset\}$ such that

$$v=\sum_{\substack{I\in\mathcal{A}\setminus\{\emptyset,(0)\}\\ \deg(I)\le m-1}} t^{\frac{\deg(I)}{2}}w_I[V_{i_1},[V_{i_2},[\cdots,V_{i_k}]\cdots](y)$$

and define

$$\Delta_{\sqrt{t}}w:=\sum_{\substack{I\in\mathcal{A}\setminus\{\emptyset,(0)\}\\ \deg(I)\le m-1}} t^{\frac{\deg(I)}{2}}w_I[e_{i_1},[e_{i_2},[\cdots,e_{i_k}]\cdots].$$

Then we take the universal weight $\pi:=\pi_{d,1}^m$ associated to w and solve consequently the above equation. This leads finally to

$$\frac{d}{d\epsilon}|_{\epsilon=0}E(f(Y_t^{y+\epsilon v}))=E(df(Y_t^y)\cdot J_{0\to t}(y)\cdot w)$$

$$=E(df(Y_t^y)\cdot J_{0\to t}(y)\cdot\sum_{k=1}^d\int_0^t J_{0\to s}(y)^{-1}V_k(Y_s^y)a_s^k ds)+\mathcal{O}(t^{\frac{m+1}{2}})$$

$$=E(\sum_{k=1}^d\int_0^t df(Y_t^y)\cdot J_{0\to t}(y)\cdot J_{0\to s}(y)^{-1}V_k(Y_s^y)a_s^k ds)+\mathcal{O}(t^{\frac{m+1}{2}})$$

$$=E(\sum_{k=1}^d\int_0^t D_s^k(Y_t^y)a_s^k ds)+\mathcal{O}(t^{\frac{m+1}{2}})$$

$$=E(f(Y_t^y)\pi)+\mathcal{O}(t^{\frac{m+1}{2}}),$$

hence the result, since due to Proposition 1 we obtain – by the Cauchy-Schwartz inequality and integration with respect to t – an order estimate of the type $\mathcal{O}(t^{\frac{m+1}{2}})$: indeed, the remainder satisfies

$$\left|E(\sum_{k=1}^d\int_0^t df(Y_t^y)\cdot J_{0\to t}(y)\cdot R_m^k(s,V_k)a_s^k ds)\right|$$

$$\le(\sum_{k=1}^d E(\int_0^t(df(Y_t^y)\cdot J_{0\to t}(y)\cdot R_m^k(s,V_k))^2ds)^{\frac12}(\sum_{i=1}^d E(\int_0^t(a_s^i)^2ds))^{\frac12}$$

$$\le\sum_{k=1}^d M(\int_0^t s^m ds)^{\frac12}=\mathcal{O}(t^{\frac{m+1}{2}}).$$

□

4. Cubature formulas for Greeks

First we have to get expressions for derivatives of heat equation evolutions on free, nilpotent Lie groups, which is an easy task for linear functions by the considerations of Section 2.

Proposition 2. *Fix a linear function λ on $\mathbb{A}^m_{d,1}$, then*

$$\frac{d}{d\epsilon}|_{\epsilon=0} E(\lambda(X_t^{x+\epsilon w_x})) = \lambda(w_x \exp(t(e_0 + \frac{1}{2}\sum_{i=1}^{d} e_i^2)))$$

for $w_x \in x(\mathfrak{g}^m_{d,1}/\langle e_0\rangle)$. Consequently

$$E(X_t^1 \pi^m_{d,1}) = \Delta_{\sqrt{t}}(w) \exp(t(e_0 + \frac{1}{2}\sum_{i=1}^{d} e_i^2)) \tag{4.1}$$

for $w \in \mathfrak{g}^m_{d,1}/\langle e_0\rangle$.

Proof. We know that the function $x \mapsto E(\lambda(X_t^x))$ solves the associated heat equation. The solution of the heat equation on $G^m_{d,1}$ with initial value λ is given by

$$x \mapsto \lambda(x \exp(t(e_0 + \frac{1}{2}\sum_{i=1}^{d} e_i^2))),$$

hence the derivative in direction w_x can be calculated and yields

$$\lambda(w_x \exp(t(e_0 + \frac{1}{2}\sum_{i=1}^{d} e_i^2))),$$

wherefrom the result follows by duality and Theorem 6. □

Next we apply Tchakaloff's theorem twice in order to obtain the appropriate cubature result.

Theorem 7. *Fix a free, nilpotent Lie group $G^m_{d,1}$ and $w \in \mathfrak{g}^m_{d,1}/\langle e_0\rangle$, then there are points $x_1, \dots, x_r \in G^m_{d,1}$ and weights $\mu_1, \dots, \mu_r \neq 0$ such that*

$$\Delta_{\sqrt{t}}(w) \exp(t(e_0 + \frac{1}{2}\sum_{i=1}^{d} e_i^2)) = \sum_{j=1}^{r} x_j \mu_j.$$

Furthermore $r \leq 2 \dim_{\mathbb{R}} \mathbb{A}^m_{d,1}$.

Proof. We write $\pi = \pi^m_{d,1}$. We define two probability measures on Wiener space, absolutely continuous with respect to Wiener measure P,

$$\frac{dQ_+}{dP} = \frac{1}{E(\pi_+)}\pi_+,$$
$$\frac{dQ_-}{dP} = \frac{1}{E(\pi_-)}\pi_-,$$

if the respective positive and negative parts have non-vanishing expectation. Then

$$E_P(X_t^e \pi) = E(\pi_+) E_{Q_+}(X_t^e) - E(\pi_-) E_{Q_-}(X_t^e).$$

There is a null set N on Wiener space, such that on N^c the process $X_t^e \in G_{d,1}^m$, hence also $X_t^e \in G_{d,1}^m$ almost surely with respect to $Q_\pm$. Consequently by Tchakaloff's theorem we find points $x_1^\pm, \dots, x_r^\pm$ and weights $\lambda_j^\pm$ such that

$$E_P(X_t^e \pi) = \sum_{j=1}^r (E(\pi_+^{(1)}) x_j^+ \lambda_j^+ - E(\pi_-^{(1)}) x_j^- \lambda_j^-),$$

which yields the desired formula by Proposition 2. □

Remark 9. *We can additionally require that* $\sum_{i=1}^r \mu_i = 0$, $\sum_{i=1}^r |\mu_i| \leq 2$ *and* $|\mu_i| \leq 1$ *for* $i = 1, \dots, r$.

Hence we can formulate a Cubature result for the calculation of Greeks.

Definition 2. *Fix* $m \geq 1$ *and* $0 < t \leq T$, *a* cubature formula for the first derivative *in direction* $\Delta_{\sqrt{t}} w \in \mathfrak{g}_{d,1}^m / \langle e_0 \rangle$ *is given by a finite number of points* $x_1, \dots, x_r \in G_{d,1}^m$ *and finitely many weights* $\mu_1, \dots, \mu_r \neq 0$, *such that*

$$E(X_t^1 \pi_{d,1}^m) = \sum_{j=1}^r \mu_j x_j,$$

or equivalently due to formula (4.1)

$$\Delta_{\sqrt{t}}(w) \exp(t(e_0 + \frac{1}{2} \sum_{i=1}^d e_i^2)) = \sum_{j=1}^r \mu_j x_j.$$

Again we use trajectories ω_i *to represent the points* x_i *as endpoints of evolutions of ordinary differential equations* $Z_t(\omega_i) = x_i$ *for* $i = 1, \dots, r$. *Furthermore* $r \leq 2 \dim_{\mathbb{R}} \mathbb{A}_{d,1}^m$.

Theorem 8. *Given* C^∞*-bounded vector fields* $V_0, \dots, V_d$ *on* $\mathbb{R}^N$, *then the diffusion process*

$$dY_t^y = \sum_{i=0}^d V^i(Y_t^y) \circ dB_t^i$$

for $y \in \mathbb{R}^N$ *admits the following cubature formulas for derivatives in direction* $v \in \mathbb{R}^N$. *Fix* $m \geq 1$ *and assume that*

$$v = \sum_{\substack{I \in \mathcal{A} \setminus (\emptyset, (0)) \\ \deg(I) \leq m-1}} t^{\frac{\deg(I)}{2}} w_I [V_{i_1}, [V_{i_2}, [\cdots, V_{i_k}] \cdots](y), \tag{4.2}$$

then we obtain

$$\frac{d}{d\epsilon}|_{\epsilon=0} E(f(Y_t^{y+\epsilon v})) = \sum_{j=1}^r \mu_j f(Y_t^y(\omega_j)) + \mathcal{O}(t^{\frac{m+1}{2}}),$$

taking a cubature formula, which was derived in $G_{d,1}^m$ *and* C^∞*-bounded* f. *The constant depends in general on derivatives of* f *up to order* $m+1$. *We can determine*

the weights μ_j and the trajectories ω_j by

$$\Delta_{\sqrt{t}}(w)\exp(t(e_0+\frac{1}{2}\sum_{i=1}^{d}e_i^2))$$
$$=\sum_{i=1}^{r}\mu_j\sum_{\substack{I\in\mathcal{A}\setminus(\emptyset,(0))\\ \deg(I)\le m}}e_{i_1}\cdots e_{i_k}\int_{0\le t_1\le\cdots\le t_k\le t}d\omega_j^{i_1}(t_1)\cdots d\omega_j^{i_k}(t_k),$$
$$\Delta_{\sqrt{t}}(w)=\sum_{\substack{I\in\mathcal{A}\setminus(\emptyset,(0))\\ \deg(I)\le m-1}}t^{\frac{\deg(I)}{2}}w_I[e_{i_1},[e_{i_2},[\cdots,e_{i_k}]\cdots].$$

Remark 10. *Since we shall in practise apply this procedure to smooth functions – as discussed in the recipe after formulas (1.3) and (1.4) – this already yields the interesting result for applications.*

Proof. By the previous constructions we obtain

$$E(X_t^x\pi_{d,1}^m)=\sum_{i=1}^{r}\mu_j\sum_{\substack{I\in\mathcal{A}\\ \deg(I)\le m}}e_{i_1}\cdots e_{i_k}\int_{0\le t_1\le\cdots\le t_k\le t}d\omega_j^{i_1}(t_1)\cdots d\omega_j^{i_k}(t_k).$$

We then insert instead of e_i the vector fields V_i and apply those vector fields to the function f at $y\in\mathbb{R}^N$. Then we obtain on the left hand side – due to stochastic Taylor expansion – and on the right hand side due to Taylor expansion of the non-autonomous equation,

$$E((f(Y_t^y)-R_m(t,f))\pi_{d,1}^m)=\sum_{i=1}^{r}\mu_j(f(Y_t^y(\omega_j))-R_{m,i}(t,f)).$$

The left hand side yields then the order estimate by partial integration and Proposition 1, the right hand side by the fact that iterated integrals with respect to ω_j behave like $t^{\frac{m+1}{2}}$. □

We additionally have an assertion on the construction of cubature paths due to the scaling properties. Assume that we found trajectories ω_i and weights μ_i for $m\ge 1$, $d\ge 1$ and $t=1$ fixed, then we can construct solutions for all $t>0$ with the same m,d: the equation

$$w\exp(e_0+\frac{1}{2}\sum_{i=1}^{d}e_i^2)=\sum_{i=1}^{r}\mu_j\sum_{\substack{I\in\mathcal{A}\\ \deg(I)\le m}}e_{i_1}\cdots e_{i_k}\int_{0\le t_1\le\cdots\le t_k\le 1}d\omega_j^{i_1}(t_1)\cdots d\omega_j^{i_k}(t_k)$$

holds, hence the trajectories

$$\eta_j^0(st):=t\omega_j^0(s),$$
$$\eta_j^i(st):=\sqrt{t}\omega_j^i(s),$$

for $i = 1, \dots, d$, $j = 1, \dots, r$ and $0 \leq s \leq 1$ satisfy

$$
\begin{aligned}
&\Delta_{\sqrt{t}}(w) \exp(t(e_0 + \frac{1}{2}\sum_{i=1}^{d} e_i^2)) \\
&= \sum_{i=1}^{r} \mu_j \sum_{\substack{I \in \mathcal{A} \\ \deg(I) \leq m}} e_{i_1} \cdots e_{i_k} \int_{0 \leq t_1 \leq \cdots \leq t_k \leq t} d\eta_j^{i_1}(t_1) \cdots d\eta_j^{i_k}(t_k),
\end{aligned}
$$

since

$$
\begin{aligned}
&\sum_{i=1}^{r} \mu_j \sum_{\substack{I \in \mathcal{A} \\ \deg(I) \leq m}} e_{i_1} \cdots e_{i_k} \int_{0 \leq t_1 \leq \cdots \leq t_k \leq t} d\eta_j^{i_1}(t_1) \cdots d\eta_j^{i_k}(t_k) \\
&= \sum_{i=1}^{r} \mu_j \sum_{\substack{I \in \mathcal{A} \\ \deg(I) \leq m}} \Delta_{\sqrt{t}}(e_{i_1} \cdots e_{i_k}) \int_{0 \leq t_1 \leq \cdots \leq t_k \leq 1} d\omega_j^{i_1}(t_1) \cdots d\omega_j^{i_k}(t_k) \\
&= \Delta_{\sqrt{t}}(\sum_{i=1}^{r} \mu_j \sum_{\substack{I \in \mathcal{A} \\ \deg(I) \leq m}} e_{i_1} \cdots e_{i_k} \int_{0 \leq t_1 \leq \cdots \leq t_k \leq 1} d\omega_j^{i_1}(t_1) \cdots d\omega_j^{i_k}(t_k)) \\
&= \Delta_{\sqrt{t}}(w \exp(e_0 + \frac{1}{2}\sum_{i=1}^{d} e_i^2)) \\
&= \Delta_{\sqrt{t}}(w) \exp(t(e_0 + \frac{1}{2}\sum_{i=1}^{d} e_i^2)).
\end{aligned}
$$

This leads to the desired assertion.

5. Applications

5.1. **Example for $m = 1$ and $d \geq 1$.** In [4] the authors provide the following expression for the Malliavin weight π: given C^∞-bounded vector fields $V_1, \dots, V_d$ on $\mathbb{R}^d$ such that a uniform ellipticity condition holds, i.e. there is $\delta > 0$ such that

$$\sum_{i=1}^{d} \langle \xi, V_i \rangle^2 \geq \delta \langle \xi, \xi \rangle$$

for all $\xi \in \mathbb{R}^d$, with respect to a standard scalar product on $\mathbb{R}^d$, the formula

$$\frac{d}{d\epsilon}|_{\epsilon=0} E(f(Y_t^{y+\epsilon v})) = E(f(Y_t^y)\frac{1}{t}\int_0^t (\sigma^{-1}(Y_s^y)(J_{0\to s}(y) \cdot v))^T dB_s), \tag{5.1}$$

where we understand dB_s as column vector, where the random matrix $\sigma^{-1}(Y_s^y)$ acts on. σ is defined via

$$\sigma(y) := (V_1(y), \dots, V_d(y)).$$

Notice that this formula provides adapted strategies, therefore the Skorohod integral can be replaced by the Ito integral. We shall compare this formula to the formula obtained in our setting. We choose $m = 1$, we can calculate $\pi_{d,1}^1$ explicitly, namely

$$\pi_{d,1}^1 = \sum_{i=1}^{d} B_t^i \frac{w_i}{\sqrt{t}}$$

for $i = 1, \dots, d$ and w has e_i-component w_i for $i = 1, \dots, d$. Hence we approximate

$$\frac{d}{d\epsilon}|_{\epsilon=0} E(f(Y_t^{y+\epsilon\sqrt{t}v})) = E(f(Y_t^y) \sum_{i=1}^{d} B_t^i \frac{w_i}{\sqrt{t}}) + \mathcal{O}(t),$$

where w_i stem from the solution of the equation

$$v = \sum_{i=1}^{d} w_i V_i(y),$$

consequently $w = \sigma^{-1}(y) \cdot v$. Hence

$$\frac{d}{d\epsilon}|_{\epsilon=0} E(f(Y_t^{y+\epsilon\sqrt{t}v})) = E(f(Y_t^y) \sum_{i=1}^{d} B_t^i \frac{\sigma^{-1}(y) \cdot v}{\sqrt{t}}) + \mathcal{O}(t),$$

$$\frac{d}{d\epsilon}|_{\epsilon=0} E(f(Y_t^{y+\epsilon v})) = E(f(Y_t^y) \sum_{i=1}^{d} B_t^i \frac{\sigma^{-1}(y) \cdot v}{t}) + \mathcal{O}(\sqrt{t}),$$

which is obviously the expansion of the precise formula (5.1), since

$$\begin{aligned} E(f(Y_t^y) \frac{1}{t} \int_0^t (\sigma^{-1}(Y_s^y)(J_{0\to s}(y) \cdot v))^T dB_s) &= E(f(Y_t^y) \sum_{i=1}^{d} B_t^i \frac{\sigma^{-1}(y) \cdot v}{t}) + \\ +E(f(Y_t^y) \frac{1}{t} \int_0^t (\sigma^{-1}(Y_s^y)(J_{0\to s}(y) \cdot v) - \sigma^{-1}(y) \cdot v)^T dB_s) \\ &= E(f(Y_t^y) \sum_{i=1}^{d} B_t^i \frac{\sigma^{-1}(y) \cdot v}{t}) + \\ +E(\frac{1}{t} \int_0^t D_s f(Y_t^y)(\sigma^{-1}(Y_s^y)(J_{0\to s}(y) \cdot v) - \sigma^{-1}(y) \cdot v)^T ds) \\ &= E(f(Y_t^y) \sum_{i=1}^{d} B_t^i \frac{\sigma^{-1}(y) \cdot v}{t}) + \mathcal{O}(\sqrt{t}) \end{aligned}$$

for f a C_b^∞-function.

5.2. **Example for $m = 2$ and $d = 2$.** In order to demonstrate the method we calculate one example of the above methodology. First we fix $m = d = 2$, hence we can include the calculations of Example 2. We first fix a direction $w = e_1$. Then we find one solution of

$$\sqrt{t}e_1 = \sqrt{t}e_1 \exp(t(e_0 + \frac{1}{2}\sum_{i=1}^{2} e_i^2)) = \sum_{i=1}^{r} \mu_i x_i,$$

which can be given through $r = 2$, $\mu_1 = \frac{1}{2}$, $\mu_2 = -\frac{1}{2}$ and $x_1 = \exp(\sqrt{t}e_1)$, $x_2 = \exp(-\sqrt{t}e_1)$. Consequently the trajectories are given by

$$\omega_1(s) = \begin{pmatrix} 0 \\ \frac{s}{\sqrt{t}} \\ 0 \end{pmatrix},$$

$$\omega_2(s) = \begin{pmatrix} 0 \\ -\frac{s}{\sqrt{t}} \\ 0 \end{pmatrix},$$

which leads to

$$Z_t^1(\omega_1) = 1 + \sqrt{t}e_1 + \frac{t}{2}e_1^2 = \exp(\sqrt{t}e_1),$$
$$Z_t^1(\omega_2) = 1 - \sqrt{t}e_1 + \frac{t}{2}e_1^2 = \exp(-\sqrt{t}e_1).$$

This in turn leads to the following approximation scheme: given vector fields V_0, V_1, V_2 on $\mathbb{R}^N$ (notice that Hörmander's hypo-ellipticity condition (1.2) is not necessarily fulfilled):

- solve the ordinary differential equations
$$dY_s^y(\omega_1) = \frac{1}{\sqrt{t}} V_1(Y_s^y(\omega_1))ds,$$
$$dY_s^y(\omega_2) = -\frac{1}{\sqrt{t}} V_1(Y_s^y(\omega_2))ds,$$
which yields flows in the diffusion direction V_1 starting at $y \in \mathbb{R}^N$.
- fix the direction for the directional derivative due to the above construction e_1 corresponds via formula (4.2) to V_1, hence $v = \sqrt{t}V_1(y)$.
- the derivative can be approximated through
$$\frac{d}{d\epsilon}|_{\epsilon=0} E(f(Y_t^{y+\epsilon\sqrt{t}V_1(y)})) = \frac{1}{2}(f(Y_t^y(\omega_1)) - f(Y_t^y(\omega_2))) + \mathcal{O}(t^{\frac{3}{2}}),$$
where the constant in the order estimate depends on the third derivative of f.

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Technical University of Vienna, e105, Wiedner Hauptstrasse 8-10, A-1040 Wien, Austria

E-mail address: jteichma@fam.tuwien.ac.at